REPORT DOCUMENTATION PAGE

AFRL-SR-BL-TR-00-

0188

0481

1. REPORT DATE (DD-MM-YYYY) 09/12/2000	2. REPORT TYPE Final	3. DATES COVERED (From - To) 03-01-1998 – 11-30-1999

4. TITLE AND SUBTITLE

Nonlinear Robust Control and Estimation

5a. CONTRACT NUMBER

5b. GRANT NUMBER
F49620-98-1-0304

5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)

Prof. William M. McEneaney

5d. PROJECT NUMBER
2304/AS

5e. TASK NUMBER
2304/AS

5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

Department of Mathematics
North Carolina State University
Raleigh, NC 27695

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES)

AFOSR/NM
801 North Randolph Street, Room 732
Arlington, VA 22203-1977

10. SPONSOR/MONITOR'S ACRONYM(S)
AFOSR

11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION / AVAILABILITY STATEMENT

May be freely distributed.

Approved for public release
distribution unlimited

13. SUPPLEMENTARY NOTES

None.

14. ABSTRACT

The overall goal has been that of computationally feasible methods for nonlinear Robust and H_∞ control and filtering, with the recent addition of risk-sensitive control and filtering. The methods are based on max-plus, min-plus and log-plus algebraic approaches to *exact* linearization of the associated semi-groups. Other tangential efforts have included making use of existing dimensional reductions in certain problems.

15. SUBJECT TERMS

Nonlinear Control, Robust Control, H_∞ Control, Estimation, Max-Plus Algebras, Numerical Method

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT	b. ABSTRACT	c. THIS PAGE			19b. TELEPHONE NUMBER (include area code)

20001016 055

DTIC QUALITY INSPECTED 4

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std. Z39.18

NONLINEAR ROBUST CONTROL AND ESTIMATION

Final Report for AFOSR Grant F49620–98–1–0304

William M. McEneaney
Department of Mathematics
North Carolina State University

1   Main Concepts:

The overall goal has been that of computationally feasible methods for nonlinear Robust and H_∞ control and filtering, with the recent addition of risk–sensitive control and filtering (due to a new result suggesting reduced computations). The methods are based on max–plus, min–plus and log–plus algebraic approaches to *exact* linearization of the associated semi–groups. Other tangential efforts have included making use of existing dimensional reductions in certain problems, and a new effort in applications to command and control for air operations.

2   Overview of Results:

1) We have begun development of max–plus methods in the solution of nonlinear H_∞ Bellman equations corresponding to problems with fixed–feedback. More specifically, we have developed error estimates and convergence results for the algorithm. Software is under development.

2) We have also addressed the nonlinear H_∞ control problem with active control computation in the case where the controller can dominate the disturbance. This leads to a semi–group which is exactly linear over the min–plus algebra. Software is under development.

3) The linearity of certain nonlinear risk–sensitive stochastic control and filtering problems under the appropriate log–plus algebra has been studied, and some initial steps indicating the nature of algorithms designed to exploit this linearity have been taken.

4) The PI (with collaborators K. Ito, Q. Zhang, and W.H. Fleming) have begun to apply methods of robust control and filtering to the C^2 problem for air operations.

5) Joint work with J.W. Helton and M.R. James on dimensional reduction for nonlinear H_∞ control under partial information in the case where there are

some very–low–noise measurements has been started. The reduction of dimension (of the space on which the information state must be propagated) under exact measurements has been demonstrated, and a simple means of compensating for the small noise in those measurements is being considered.

6) Another, very interesting, development is that nonlinear risk–sensitive control and nonlinear H_∞ control all take exactly the form of standard stochastic control (i.e. minimization of the expectation of an integral cost) under changes of algebra and probability measure. This work is in a nascent state.

7) Joint work with M.R. James on application of max–plus methods to nonlinear H_∞ control *under partial information* is underway. Software has been developed by E. Gallestey (a post-doc) at Australian National University.

8) Joint work with W.H. Fleming proving convergence of the risk-sensitive filter to the H_∞ filter (McEneaney definition) has reached a mature stage. This not only proves a filtering result analogous to the now well–known control result, but also lends additional support to the use of the computationally preferable H_∞ filter definition of McEneaney (in contrast to definitions used by some other researchers which are actually more directly analogous to smoothers in stochastic approaches).

3 Detailed Discussions

3.1 Numerics – Overview of Max–Plus Methods

We focused on the nonlinear H_∞ control problem during the grant period. Consider the case where one has chosen a particular feedback form. Then the associated semi–group is linear over the max–plus algebra. This led to a representation of the value function (available storage) as a max–plus eigenvector of the semi–group corresponding to eigenvalue zero (the max–plus *multiplicative* identity), that is $0 \otimes W = \mathcal{S}_\tau[W]$ where W is the value, $\mathcal{S}_\tau$ is the semi–group, and $\otimes$ represents max–plus multiplication. By approximating W with a finite number of functions from a max–plus basis expansion, the problem is reduced to computation of a finite–dimensional max–plus eigenvector, e, satisfying $0 \otimes e = B \otimes e$ for a particular B.

More recently, we began developing an error analysis which demonstrates that the errors converge to zero as the number of vectors used in the basis expansion increases. It is known that there is a unique max–plus eigenvalue for the class of matrices to which B belongs. We have now demonstrated that (in this class of matrices), there is also a unique eigenvector corresponding to this unique eigenvalue. We use the power method to compute this eigenvector, and guarantee exact convergence in a

finite number of steps. This quality of a unique eigenvector removes some previous questions as to whether the method would converge to the correct solution.

More specifically, we consider the H$_\infty$ problem for a nonlinear system. Nonlinear H$_\infty$ control and its relation to the associated dynamic programming equation (DPE) has been studied extensively (see [1], [11], [42], [3] among many notable others). For continuous systems, this DPE takes the form of a partial differential equation (PDE). Unfortunately, the H$_\infty$ DPE is nonlinear – possessing a term which is quadratic in the gradient. A significant recent development has been the discovery of new numerical methods exploiting the max–plus (to be defined below) linearity of the associated solution operator [18], [19], [17], [8]. These algorithms employ a max-plus basis expansion for the space of semi–convex functions. The max–plus linearity was also noticed earlier in [12]. In this approach, the solution of the H$_\infty$ PDE is given by a max–plus eigenvector corresponding to a max–plus eigenvalue of 0 (the max–plus multiplicative identity) for a particular matrix. This matrix is associated with the solution operator of the PDE. Note that the nonlinear H$_\infty$ problem is transformed into a *max–plus eigenvector computation* although one must still compute (or approximate) the matrix defining this problem. This algorithm is analogous to a spectral method for a linear problem (but in the max–plus sense) as opposed to finite difference methods. We remind the reader that the max–plus algebra is a commutative semi–field on $\Re \cup \{-\infty\}$ given by

$$a \oplus b = \max\{a, b\},$$
$$a \otimes b = a + b \tag{1}$$

Note that $-\infty$ is the additive identity, and 0 is the multiplicative identity. This can be extended to a field [2], but we do not need that here.

3.2   Details – Solution operator and max–plus linearity

Consider the system

$$\dot{X} = f(X) + \sigma(X)w, \qquad X(0) = x \tag{2}$$

where X is the state taking values in $\Re^n$, f represents the nominal dynamics, the disturbance w lies in $\mathcal{W} \doteq \{w : [0, \infty) \to \Re^m : w \in L_2[0, T] \; \forall T < \infty\}$, and σ is an $n \times m$ matrix–valued multiplier on the disturbance.

We will make the following assumptions. These assumptions are probably not necessary but are sufficient for the results to follow. We will assume that all the functions f, σ and l (given below) are smooth. We will assume that there exist $K, c \in (0, \infty)$ such that

$$|f(x) - f(y)| \le K|x - y| \qquad \forall\, x, y \in \Re^n$$
$$(x - y)^T (f(x) - f(y)) \le -c|x - y|^2 \qquad \forall\, x, y \in \Re^n \tag{A1}$$
$$f(0) = 0$$

Note that the second inequality automatically implies the closed–loop stability criterion of H_∞ control. (The second inequality would not be needed in the active control case to follow.) We assume that

$$\begin{aligned} |\sigma(x) - \sigma(y)| &\leq K|x - y| \qquad \forall\, x, y \in \Re^n \\ |\sigma(x)| &\leq M \qquad \forall\, x \in \Re^n \end{aligned} \qquad (A2)$$

for some $M < \infty$. Let $l(x)$ be the running cost. We assume that there exist $\beta, \alpha < \infty$ such that

$$\begin{aligned} |l_{xx}(x)| &\leq \beta \qquad \forall\, x \in \Re^n \\ 0 \leq l(x) &\leq \alpha|x|^2 \qquad \forall\, x \in \Re^n. \end{aligned} \qquad (A3)$$

(There is a reason for allowing β to be greater than 2α, which one might notice below.)

The system is said to satisfy an H_∞ attenuation bound (of γ) if there exists $\gamma < \infty$ and a locally bounded available storage function $W(x)$, such that

$$W(x) = \sup_{w \in \mathcal{W}} \sup_{T < \infty} \int_0^T l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2\, dt. \qquad (3)$$

The corresponding DPE is

$$0 = \frac{1}{2\gamma^2}\nabla W^T \sigma(x)\sigma^T(x)\nabla W + f^T(x)\nabla W + l(x). \qquad (4)$$

It will be assumed throughout that we are looking for a solution satisfying $W(0) = 0$. We will also suppose that the above constants satisfy

$$\frac{\gamma^2}{2M^2} > \frac{\alpha}{c^2}. \qquad (5)$$

We note that there are linear examples where (5) exactly determines the minimum disturbance attenuation parameter. One has the following result ([26],[30]). The proof can be found in the references.

Rather than dwelling on the PDE representation (4), we would like to show that W is a fixed point of the solution operator to the DPE, where this solution operator has the representation

$$S_\tau[W(\cdot)](x) = \sup_{w \in \mathcal{W}}\Big\{\int_0^\tau l(X(t)) \frac{\gamma^2}{2}|w(t)|^2\, dt \\ + W(X(\tau))\Big\}.$$

By a modification of the proof of the previous theorem, one can show the following. The proof can be found in [19].

Theorem 3.1 *For any $\tau \in [0, \infty)$, W given by (3) satisfies $S_\tau[W] = W$, and further, it is the unique solution in the class*

$$0 \leq W(x) \leq c \frac{(\gamma - \delta)^2}{2M^2} |x|^2 \tag{6}$$

for some $\delta > 0$, and this is the "correct" solution given by (3).

Note that W is a fixed point of S_τ for any τ, which provides some freedom in the choice of problem we wish to solve. This will be discussed further below. The proof of the max-plus linearity of S_τ is trivial (see [19]).

Theorem 3.2 *The solution operator, S_τ, is linear in the max-plus algebra. That is, (for constant c and functions ϕ, ψ)*

$$S_\tau[\phi \oplus \psi](x) = S_\tau[\phi](x) \oplus S_\tau[\psi](x)$$
$$S_\tau[c \otimes \phi](x) = c \otimes S_\tau[\phi](x)$$

3.3 Details – Max–plus basis and the eigenvector equation

Define the space of semi-convex functions, $\mathcal{S}$, as the set of $\psi : \Re^n \to \Re$ such that for any $R < \infty$ there exists $c_R < \infty$ such that $\psi(x) + \frac{c_R}{2}|x|^2$ is convex over $B_R = \{x \in \Re^n : |x| \leq R\}$ (see [18]). We refer to such a c_R as a semi-convexity constant for ψ. The proof of the following is rather technical.

Theorem 3.3 *W lies in $\mathcal{S}$.*

The following max-plus basis can be derived from convex duality; see [18] for details. Let $\phi \in \mathcal{S}$. Fix $R < \infty$. Then ϕ is Lipschitz continuous with some constant L_R over $B_R(0)$ (cf.[6]). Let $\{x_i\}$ be a countable, dense set over $B_{L_R/(2c_R)}(0)$, and let $C > c_R I$ (see [18] for definition) be a positive definite matrix. Define

$$\psi_i(x) \doteq -\frac{1}{2}(x - x_i)^T C (x - x_i)$$

for each i. Then,

$$\phi(x) = \bigoplus_{i=1}^{\infty} [a_i \otimes \psi_i(x)] \qquad \forall x \in B_R \tag{7}$$

where $a_i \doteq -\max_{x \in B_R}[\psi_i(x) - \phi(x)]$. This is a max-plus basis expansion.

We assume C can be chosen so that $S_\tau(\psi_i(x))$ has an expansion

$$S_\tau(\psi_i(x)) = \bigoplus_{i=1}^{\infty} B_{j,i} \otimes \psi_j \tag{A4}$$

where

$$B_{j,i} = -\max_{x \in B_R}(\psi_j(x) - S_\tau(\psi_i(x)))$$

for each i (we discuss this a little more in the next section). (This is equivalent to an assumption that $S_\tau(\psi_i(x))$ is semi-convex with some constant c_R' on B_R where $C > c_R'I$. In the case where $\sigma\sigma^T$ is uniformly nondegenerate, the existence of c_R' can be proven, but we do not include that here.) Note that B actually depends on τ, but for this section we fix any value τ, and suppress the dependence in the notation. In order to reduce complexity, we suppose throughout the next two sections that W has a max–plus basis expansion with a finite number of terms. Let $W(x) = \bigoplus_{i=1}^{n} a_i \otimes \psi_i$ and $a^T = (a_1 \ a_2 \cdots a_n)$. Also assume that the expansions of the $S_\tau(\psi_i(x))$ terminate at n terms. The full error analysis (where we show that the errors introduced by truncation go to 0 as $n \to \infty$) will be delayed to a later paper. Lastly, let us assume that it is required that

$$a_j > -\infty \qquad \forall j \leq n, \tag{A5}$$

that is, each basis function must be active. The key result is Theorem 3.4. The proof appears in [17].

Theorem 3.4 $S_\tau[W] = W$ *if and only if* $a = B \otimes a$ *where* $B \otimes a$ *represents max–plus matrix multiplication.*

The proof appears in [13].

3.4 Details – Convergence of the eigenvector computations

Recall that we need to solve the eigenvector problem

$$0 \otimes e = B \otimes e, \quad \text{or}, \quad e = B \otimes e. \tag{8}$$

This requires two steps: computing (approximately) B, and then solving (8) given this B. In this section, we address the second step; in the next section, we will address the first step.

We will use the power method to compute the eigenvector. That is, we will demonstrate that

$$e = \lim_{m \to \infty} B^m \otimes 0$$

where 0 represents the zero vector, and the B^m represents the max–plus product repeated m times.

For the remainder of the section, fix any $\tau \in (0, \infty)$. Define

$$H(x,y) \doteq \mathcal{S}_\tau[W](x) - \sup_{w \in \mathcal{W}_y} \int_0^\tau l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2 \, dt$$
$$+ W(X(\tau))\} \tag{9}$$

where $X(0) = 0$ and $\mathcal{W}_y = \{w \in \mathcal{W} : X(\tau) = y\}$. We note the following, but do not include the straightforward proof (although the machinery is in [30])

$$H(\cdot, \cdot) \quad \text{is continuous,} \tag{10}$$
$$H(0,0) = 0 \quad \text{and} \quad H(x,x) > 0 \text{ if } x \neq 0. \tag{11}$$

Lemma 3.5 *Let $w \in \mathcal{W}$.*

$$\int_0^\tau l(X(t)) - \frac{\gamma^2}{2}|w(t)|^2 \, dt \leq W(x)W(X(\tau))$$
$$- H(x, X(\tau))$$

where $X(0) = x$.

The proof appears in [13].

Now let the $\{x_j\}$ be such that $x_1 = 0$.

Lemma 3.6 *$B_{1,1} = 0$. Also, there exists $\delta > 0$ such that for all $j \neq 1$, $B_{j,j} \leq -\delta$.*

The proof appears in [13].

Theorem 3.7 *Let $N \in \mathcal{N}$, $\{k_i\}_{i=1}^{i=N+1}$ such that $1 \leq k_i \leq n$ for all i and $k_{N+1} = k_1$. Suppose we are not in the case $N = 1$, $k_1 = k_2 = 1$. Then*

$$\sum_{i=1}^{N} B_{k_i, k_{i+1}} \leq -\delta.$$

The proof appears in [13].

We should note that B has a unique eigenvalue, although possibly many eigenvectors corresponding to that eigenvalue [2]. By the above results, this eigenvalue must be zero (ignoring errors due to approximation).

Theorem 3.8 *$\lim_{N \to \infty} B^N \otimes 0$ exists, converges in a finite number of steps, and satisfies $e = B \otimes e$.*

The proof appears in [13].

Not only is the eigenvalue unique, but we can also show that

Corollary 3.9 *There is a unique eigenvector up to a max-plus multiplicative constant, and of course, this is the output of the above power method.*

The proof appears in [13].

3.5   Details – Approximation of B

The feasibility of the algorithm is dependent upon a feasibile approximation algorithm for B. One approach is a Taylor series (in t) approximation to $S_t[\psi_i](x)$. More specifically, letting $V(t,x) = S_t[\psi_i](x)$, so that V satisfies

$$V_t = f \cdot \nabla V + l + \nabla V^T \sigma \sigma^T \nabla V \tag{12}$$
$$V(0,x) = \psi_i(x)$$

one may approximate V as

$$V(t,x) = V_0(x) + V_1(x)t + \frac{1}{2}V_2(x)t^2 + \ldots \tag{13}$$

Here $V_0(x) = \psi_i(x)$ and V_1 is the right hand side of (12) with ψ_i replacing V. The higher order terms are computed by differentiating (12), and we do not include them. Then

$$B_{j,i} = -\max_x \left\{ \psi_j(x) - [V_0 + V_1\tau + \frac{1}{2}V_2\tau^2 + \ldots](x) \right\}.$$

This method was applied but suffers from a problem which we describe only briefly. Note that the approximation of V via, say three terms, in the Taylor series at x out to time τ can be improved by reducing τ. However, the argmax moves off toward "∞" as $\tau \downarrow 0$, and the Taylor series approximation degrades as x moves off to ∞! Consequently, the approximation of $B_{j,i}$ does not tend to improve as $\tau \downarrow 0$, and so this approach was abandoned.

In its stead we are using a means of approximately tracking the argmax for small time intervals. Let

$$\overline{\psi}_j(t,x) = -\frac{1}{2}(x - \xi(t))^T C(x - \xi(t))$$

where

$$\xi(t) = x_i + (t/\tau)(x_j - x_i).$$

Let $V(t,x)$ be as above (with ψ_i as initial condition). Let

$$\overline{X}(t) = \operatorname{argmax}\left\{ \overline{\psi}_j(t,x) - V(t,x) \right\}.$$

Note that

$$\overline{X}(\tau) = \operatorname{argmax}\left\{ \psi_j(x) - V(\tau,x) \right\}$$

which is the desired quantity. The replacement of $\psi_j(\cdot)$ by $\overline{\psi}_j(t,\cdot)$ prevents the argmax from "blowing up" at $t \downarrow 0$. We use the Taylor expansions

$$V(t,x) \approx V_0(t) + V_1(t)(x - \overline{X}(t))$$
$$+ \frac{1}{2}(x - \overline{X}(t))^T V_2(t)(x - \overline{X}(t)) + \ldots$$
$$f_k(x) \approx f_{k0} + f_{k1}(x - \overline{X}(t)) + \ldots$$

(where the k subscripts in the f expansion indicate components). One then obtains the following sequence of ordinary differential equations for the propagation of $\overline{X}$ leading to a computation of $B_{j,i}$:

$$\dot{\overline{X}}_k = (V_2 - C)^{-1}\Big\{\frac{1}{\tau}\sum_l C_{k,l}(x_{il} - x_{jl})$$

$$-\Big[\sum_l((f_{l1})_k V_{1x_l} + f_{l0}V_{2x_k,x_l})$$

$$-2\sum_{l,m}(V_{1x_l}a_{l,m}V_{2x_l,x_k}) + 2\overline{X}_k\Big]\Big\}$$

$$\dot{V}_0 = \sum_k\Big[\overline{\psi}_{j_{x_k}}(f_{k0} + \overline{X}_k - \sum_l(a_{k,l}\overline{\psi}_{j_{x_l}}))\Big] + |\overline{X}|^2$$

plus higher order equations which we do not include here. It will be shown that the error induced by cutting off this series at a finite number of differential equations can be bounded for small time. We also note that the initial conditions for this propagation are easily obtained, although we do not include them here.

Two examples appear in figures 1 and 2. In the first, a linear–quadratic example is considered. Both the exact solution, and the numerically computed solution are displayed. In figure 2, the solution of a nonlinear problem is displayed. Although there is a significant nonlinearity, the solution is still rather well–behaved; less well–behaved examples will need to be generated. Each computation took less than 10 seconds on a Unix Ultra 10 workstation.

There still remain some significant questions, such as how to reasonably choose the basis functions and time–step.

Further information can be found in [13], [17], [19] and [18].

3.6   Numerics – Min–Plus Methods

Min–plus methods are being used for the solution of nonlinear H_∞ control problems where there is active control computation (as opposed to testing of feedbacks above). For instance, consider the problem with dynamics $\dot{X} = A(X) + B(X)u + \sigma(X)w$ and cost

$$J = \int_0^\infty L(X(t)) + \frac{\eta^2}{2}|u(t)|^2 - \frac{\gamma^2}{2}|w(t)|^2\, dt$$

where u is the control, w is the disturbance and X is the state. The corresponding Isaacs equation takes the form

$$0 = A(x)\cdot\nabla W + L(x) - \nabla W^T\left[\frac{B(x)B^T(x)}{2\eta^2} - \frac{\sigma(x)\sigma^T(x)}{2\gamma^2}\right]\nabla W.$$

If u dominates w in the sense that the Hamiltonian is concave, then the associated semi–group is linear over the min–plus algebra. All the results of item 1 have analogues in this case. Software for two–dimensional problems is being debugged.

Further information can be found in [13] and [17].

3.7 Min–Plus Example

As an example, we consider the min–plus problem in two–dimensions with $X(t) = (x_1(t), x_2(t))^T$ and dynamics

$$\begin{pmatrix} \dot{x}_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ (3/4) \arctan(2x_1) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w. \tag{14}$$

Let γ, η be such that the reduction of the previous section yields

$$\frac{QQ^T}{2} = \begin{bmatrix} 0 & 0 \\ 0 & 1/2 \end{bmatrix}.$$

Note that this example was chose so as to represent a second–order system of the form

$$\ddot{y} = (3/4) \arctan(2y) + u + w.$$

The running cost was simply $l(X) = x_1^2 + x_2^2$.

The computations were run to obtain the results depicted in the figures below. The computations were run at a coarseness level that allowed computation in less than 10 seconds on a Sun Ultra 10. Figure 1 depicts W. Figures 2 and 3 depict the partials of W; note the non–smoothness. Figure 4 depicts the approximate error computed by taking first–order differences (probably too coarse a method) on the grid to approximate ∇W, and then plugging this back into the PDE. The resulting values were scaled by dividing by $1 + |x|^2$.
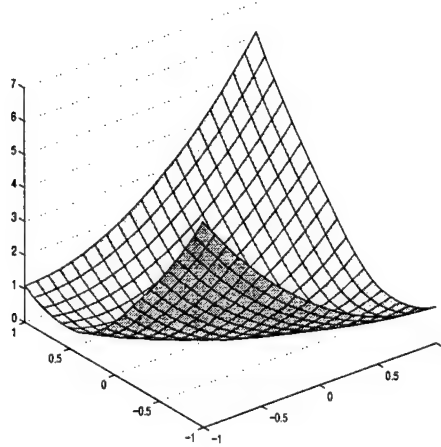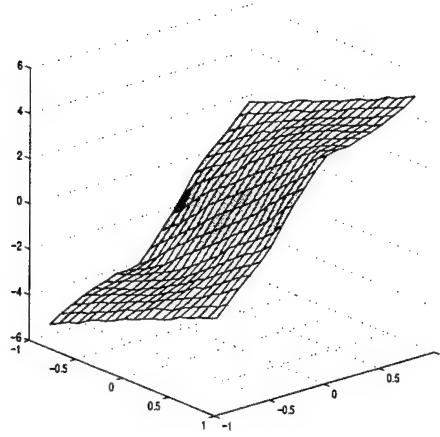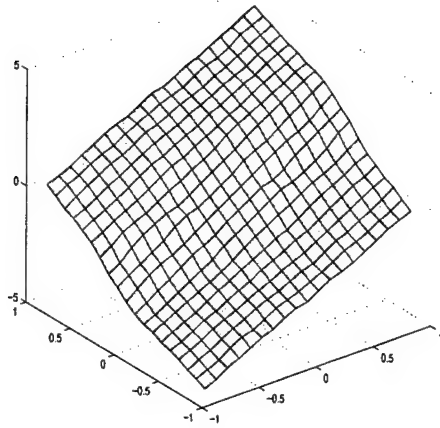

Figure 1: Storage function

Figure 2: Partial

Figure 3: Partial

3.8 Numerics – Log–Plus Methods

Nonlinear risk–sensitive problems such as risk–sensitive filtering exhibit the same type of behavior as described in the first item in the case of H_∞ problems. Specifically, there exist a set of algebras (actually fields in this case) indexed by a parameter ε, such that the risk–sensitive problems have semi–groups which are linear. The question of how to choose an appropriate basis to exploit this linearity remains open.

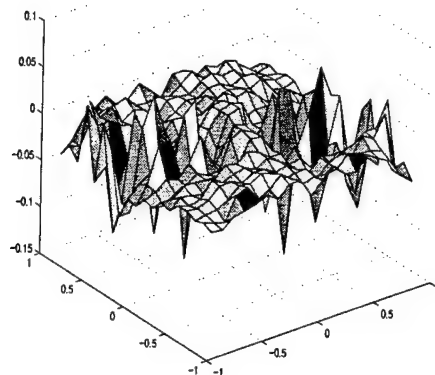Further documentation can be found in [16].

Figure 4: Scaled back-substitution error

3.9 Theory/Numerics – Dimensional Reduction

Since filtering and control under partial observations can quickly become computationally unwieldy as the dimension increases, we consider the case where the measurements are such that one might know *some* components of the state almost exactly. This has been termed the "cheap sensor" case. We showed that if m state–components were measured exactly in a problem with n state–components, the computation of the information state could be reduced to propagation on an affine $n - m$ dimensional space, with this affine space moving according to the exact measurements. This was been documented in a CDC paper, and a journal paper has been submitted to IEEE TAC. However, one needs to ensure that the method is robust to small errors in the nearly exact measurements. We are now beginning a study of this problem. We will show that a "steep" quadratic in the m state components will be the next step in an expansion around the exact measurement situation. (Joint work with J.W. Helton.)

Further documentation can be found in [10] and [20].

3.10 Theory – Risk–Sensitive Filtering

It has been demonstrated that a risk–sensitive filter converges to a Robust/H_∞ filter (McEneaney definition). This result is directly analogous to similar results which we have obtained in recent years for the control problem. This also adds further weight to the appropriateness of the (McEneaney definition of the) H_∞ filter *for tracking applications* (as opposed to more computationally difficult definitions being employed for partially observed control which are actually more properly termed

smoothers) (Joint work with W.H. Fleming.)

Further documentation can be found in [7] and [24].

3.11   Theory/Numerics – Partially Observed H$_\infty$ Control

In joint work with M.R. James, we have developed an approach to H$_\infty$ control under partial information where the information state is propagated via max–plus methods. Software has been developed at Australia National University with the aid of a post–doc.

Further documentation can be found in [15].

3.12   Theory – General

We showed that nonlinear risk–sensitive problems and H$_\infty$ problems take exactly the same form as traditional stochastic problems – once one transforms the underlying algebra. That is, in traditional stochastic control problems, one attempts to minimize an expectation of some integral cost criterion such as $E\{\int_0^T L(X(t))\,dt\}$. A corresponding risk–sensitive criterion would take the form

$$\varepsilon \log \left[E \left\{ \exp \left(\frac{1}{\varepsilon} \int_0^T L(X(t))\,dt \right) \right\} \right], \tag{2}$$

while an H$_\infty$ criterion would take the form

$$\sup_{w \in L^2} \int_0^T [L(X(t)) - |w(t)|^2]\,dt. \tag{3}$$

Now, employ the appropriate log–plus algebra for criterion (2), and the max–plus algebra for criterion (3). Next, adjust the definition of a probability measure to coincide with the algebra. For instance, the probability measure, P', must satisfy $I_a \leq P'(A) \leq I_m$ for any measurable A where I_a and I_m are the additive and multiplicative identities. (In the standard algebra, this is simply $0 \leq P(A) \leq 1$.) Then criteria (2), (3) take the (traditional stochastic) form

$$E^\varepsilon \left\{ \int_0^T L(X(t))\,dt \right\}.$$

where the ε superscript indicates a probability measure corresponding to the appropriate log–plus or max–plus algebra. As $\varepsilon \downarrow 0$, the log–plus algebras converge to the max–plus algebra, and the log–plus probability measures converge to the max–plus measure. Consequently, one can also interpret the convergence of risk–sensitive control to H$_\infty$ control in terms of convergence of the algebras and corresponding probability measures. This is a major re–interpretation. This work is in its infancy.

Further documentation can be found in [16].

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